

ADDITIONAL SEPARATION AXIOMS FOR SYNTOPOGENOUS SPACES

BY

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(Communicated by Prof. H. FREUDENTHAL at the meeting of September 26, 1964)

Introduction

The T_0 , T_1 , and T_2 separation axioms for syntopogenous spaces were considered by Á. CSÁSZÁR [1, 2]. In our earlier paper [7] we considered the regular, completely regular, normal, and completely normal separation axioms for syntopogenous spaces. The purpose of this paper is to discuss the Stone and Urysohn separation axioms in these general spaces. We will assume the reader is somewhat familiar with the notation and results in both Császár's book and our earlier paper.

1. *Definitions and elementary properties*

Let $[E, \mathcal{S}]$ denote an arbitrary syntopogenous space. It will be called an \mathcal{S} -Urysohn space if for $x \neq y$, x and y have neighborhoods whose closures are disjoint. The syntopogenous space $[E, \mathcal{S}]$ will be called an \mathcal{S} -Stone space if for $x \neq y$, there exists an $(\mathcal{S}, \mathcal{H})$ -continuous function f of E into R (the reals with the usual uniformity \mathcal{H}) such that $f(x)=0$ and $f(y)=1$. If the syntopogenous structure \mathcal{S} is simple and perfect, one sees that the above definitions are the usual topological definitions. Using the terminology of [6], a syntopogenous space $[E, \mathcal{S}]$ is an \mathcal{S} -Urysohn space iff for $x \neq y$, x and y have neighborhoods whose closures are \mathcal{S} -separated.

The definitions clearly show that every \mathcal{S} -Urysohn space is an $\mathcal{S}-T_2$ space. To show that every \mathcal{S} -Stone space is an \mathcal{S} -Urysohn space, let $x \neq y$ in E . Then there exists an $(\mathcal{S}, \mathcal{H})$ -continuous function f of E into R such that $f(x)=0$ and $f(y)=1$. The sets $f^{-1}([-1/4, 1/4])$ and $f^{-1}([3/4, 5/4])$ are the required \mathcal{S} -separated sets. Every $\mathcal{S}-T_3'$ space is an \mathcal{S} -Stone space since $x \neq y$ implies $x \notin c(\{y\})$. Thus the same implications hold here as hold between the separation axioms for topological spaces.

E. HEWITT [4] gave topological examples of a T_2 space which is not a Urysohn space and of a Urysohn space which is not a Stone space.

*) Both authors were supported in part by the National Science Foundation under research grant NSF G-22690 and the first author by a National Science Foundation Research Participation Award.

An example of a Urysohn Stone space which is not a T_3 space can be found in [5; p. 61] and an example of a T_3 space which is not a Stone space is found in [3]. This shows that no additional implications hold, even in topological spaces.

2. Additional properties

The definition of an \mathcal{S} -Urysohn space depends only on neighborhoods of points and closures of sets. Since both of these depend only on the topology (\mathcal{S}^{tp}) generated by the syntopogenous structure \mathcal{S} , we see that this definition is purely topological, as were the T_0 , T_1 , and T_2 axioms of Császár. The example in [7] shows that the \mathcal{S} -Stone axiom is not purely topological since the space $[E, \mathcal{S}^{tp}]$ is \mathcal{S}^{tp} -Stone but the space $[E, \mathcal{S}]$ is not \mathcal{S} -Stone. The following lemma proves that the \mathcal{S} -Stone axiom for a syntopogenous space $[E, \mathcal{S}]$ depends entirely on the \mathcal{S}^t structure.

Lemma 1. $[E, \mathcal{S}]$ is an \mathcal{S} -Stone space iff $[E, \mathcal{S}^t]$ is an \mathcal{S}^t -Stone space.

Proof. The function f in the definition may be restricted to be into $I = [0, 1]$ with the uniformity $\mathcal{H}|I$. For any topogenous structure \mathcal{T} , let \mathcal{T}^* denote the least fine syntopogenous structure \mathcal{S} such that $\mathcal{S}^t = \mathcal{T}$ (see [1; (13.53)]). Then every $(\mathcal{S}^t, \mathcal{H}|I)$ -continuous function f is $(\mathcal{S}^t, (\mathcal{H}|I)^t)$ -continuous [1; (10.12)], $(\mathcal{S}^{t*}, (\mathcal{H}|I)^{t*})$ -continuous [7; Lemma 1], and $(\mathcal{S}, \mathcal{H}|I)$ -continuous [1; (10.10)] since $(\mathcal{H}|I)^{t*} \sim \mathcal{H}|I$ [1; (15.58)] and $\mathcal{S}^{t*} < \mathcal{S}$ by construction. Conversely, if f is $(\mathcal{S}, \mathcal{H}|I)$ -continuous, then f is $(\mathcal{S}^t, \mathcal{H}|I)$ -continuous [1; (10.10)].

We can see from the above paragraph that the \mathcal{S} -Urysohn axiom is invariant under homeomorphisms while the \mathcal{S} -Stone axiom is only invariant under equimorphisms. The \mathcal{S} -Urysohn axiom is hereditary since disjoint neighborhoods induce disjoint neighborhoods in any subspace. If A is a subspace of $[E, \mathcal{S}]$, then every $(\mathcal{S}, \mathcal{H})$ -continuous function of E into R induces an $(\mathcal{S}|A, \mathcal{H})$ -continuous function of A into R [1; (10.20)]. This implies that the \mathcal{S} -Stone axiom is also hereditary.

Let E be the product of the sets $\{E^\lambda : \lambda \in A\}$ and \mathcal{S} be the product syntopogenous structure of the $\{\mathcal{S}^\lambda : \lambda \in A\}$ where $\{[E^\lambda, \mathcal{S}^\lambda] : \lambda \in A\}$ are given syntopogenous spaces.

Lemma 2. The space $[E, \mathcal{S}]$ is \mathcal{S} -Urysohn iff each space $[E^\lambda, \mathcal{S}^\lambda]$ is \mathcal{S}^λ -Urysohn.

Proof. The necessity follows from the fact that the axiom is hereditary and preserved by homeomorphisms. Suppose each $[E^\lambda, \mathcal{S}^\lambda]$ is \mathcal{S}^λ -Urysohn and let $x \neq y$ in E . Then there is $\lambda \in A$ such that $x^\lambda = pr_\lambda(x) \neq pr_\lambda(y) = y^\lambda$. But $x^\lambda \neq y^\lambda$ in E^λ implies there exist \mathcal{S}^λ -separated neighborhoods A^λ and B^λ of x^λ and y^λ . The sets $pr_\lambda^{-1}(A^\lambda)$ and $pr_\lambda^{-1}(B^\lambda)$ are the necessary \mathcal{S} -separated neighborhoods of x and y .

Lemma 3. The space $[E, \mathcal{S}]$ is \mathcal{S} -Stone iff each space $[E^\lambda, \mathcal{S}^\lambda]$ is \mathcal{S}^λ -Stone.

Proof. Again the necessity follows from the previous properties. Suppose each $[E^\lambda, \mathcal{S}^\lambda]$ is \mathcal{S}^λ -Stone and let $x \neq y$ in E . Then there exists $\lambda \in \Lambda$ such that $x^\lambda = pr_\lambda(x) \neq pr_\lambda(y) = y^\lambda$. But $x^\lambda \neq y^\lambda$ in E^λ implies there is an $(\mathcal{S}^\lambda, \mathcal{H})$ -continuous function f of E^λ into R such that $f(x^\lambda) = 0$ and $f(y^\lambda) = 1$. Then the function $g(z) = f(pr_\lambda(z))$ is the required $(\mathcal{S}, \mathcal{H})$ -continuous function of E into R and so $[E, \mathcal{S}]$ is \mathcal{S} -Stone.

The above lemmas show that both the \mathcal{S} -Urysohn and \mathcal{S} -Stone axioms are invariant under arbitrary products. Thus we have shown that most of the usual topological properties hold in our general spaces.

3. Applications

Since every symmetric syntopogenous space is \mathcal{S} -completely regular (see [7]), one sees that both the \mathcal{S} -Urysohn and the \mathcal{S} -Stone axioms are equivalent to the $\mathcal{S}-T_1$ and $\mathcal{S}-T_2$ axioms in symmetric syntopogenous spaces. Therefore this does not add anything to the theory of uniform or proximity spaces but does add to the theory of quasi-uniform and non-symmetrical proximity spaces.

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